Decimal Determination - Solution

First, we consider an example. For $A = B = 1$ and $s = 3$, the corresponding decimal expansion is

$$\frac{1}{1 - 0.00010001} = 1.00010001000200030005000800130021003400550089014402 33\ldots.$$

The key is to notice the Fibonacci numbers in this expansion. There are blocks of $s+1$ digits consisting of Fibonacci numbers, potentially padded with zeroes in front. The key to prove this, and also the key to solving the problem for general A and B, is to look at the generating function for the Fibonacci numbers. That is, to find a function $f(x)$ with series expansion

$$f(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \ldots = \sum_{n=0}^{\infty} F_{n+1}x^n$$

If we use the recursion for Fibonacci numbers, we can derive a functional equation on $f(x)$:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} F_{n+1}x^n \\
&= 1 + \sum_{n=1}^{\infty} F_{n+1}x^n \\
&= 1 + \sum_{n=1}^{\infty} \left(F_n + F_{n-1}\right) x^n \\
&= 1 + \sum_{n=1}^{\infty} F_n x^n + \sum_{n=1}^{\infty} F_{n-1}x^n \\
&= 1 + \sum_{n=0}^{\infty} F_{n+1}x^{n+1} + \sum_{n=-1}^{\infty} F_{n+1}x^{n+2} \\
&= 1 + \sum_{n=0}^{\infty} F_{n+1}x^{n+1} + \sum_{n=0}^{\infty} F_{n+1}x^{n+2} + F_0 x \\
&= 1 + x\sum_{n=0}^{\infty} F_{n+1}x^n + x^2\sum_{n=0}^{\infty} F_{n+1}x^n + 0 \\
&= 1 + xf(x) + x^2 f(x).
\end{aligned}$$

Hence, $f(x) = 1/(1 - x - x^2)$. Note that if we substitute $x = 10^{-(s+1)}$, this formula is exactly $D(1,1,s)$. Then $D(1,1,s) = \sum_{n=0}^{\infty} F_{n+1}10^{-k(s+1)}$, which precisely explains the emerging Fibonacci numbers in the decimal expansion.

We can now generalize to $D(A,B,s)$ by reversing and slightly modifying the above argumentation for $D(1,1,s)$. Thus, we instead look for the series expansion of $f(x) = 1/(1-Ax-Bx^2)$. If we let $f(x) = \sum_{n=0}^{\infty} a_{n+1}x^n$, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+1}x^n &= f(x) \\
&= 1 + Axf(x) + Bx^2 f(x) \\
&= 1 + A\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + B\sum_{n=0}^{\infty} a_{n+1}x^{n+2} \\
&= 1 + A\sum_{n=1}^{\infty} a_n x^n + B\sum_{n=2}^{\infty} a_{n-1}x^n \\
&= 1 + Aa_1 x + \sum_{n=2}^{\infty} \left(Aa_n + Ba_{n-1}\right) x^n.
\end{aligned}$$

By looking at each coefficient separately, we derive that $a_1 = 1, a_2 = Aa_1 = A$ and $a_{n+1} = Aa_n + Ba_{n-1}$ for $n \geq 2$. Hence, we have a similar recursive structure in the expansion for $D(A,B,s)$. Also, one can show that $a_{n+1} \leq 10^n$ for any $A, B \in \{0,1,\ldots,9\}$ by induction (left as an exercise). As the decimal expansion consists of blocks of $s+1$ digits ending in a single a_n, as long as we do not go past a_{s+1}, there will be no overlap between a_n from different blocks. Luckily, as $d \leq s(s+1)$, this works out perfectly.

The resulting algorithm works as follows. First, determine which block of $s+1$ digits should be looked at by computing $(d-1)//(s+1)$. This corresponds to an element a_n. Calculate this element simply by recursing. There are faster ways to recurse, but the direct formula will not work due to floating point rounding errors. Then select the correct digit of a_n and return this.